Emergence of linear wave segments and predictable traits in saturated nonlinear media

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We find the key behind the existence traits of asymptotic saturated nonlinear optical solitons in the emergence of linear wave segments. These traits, produced by the progressive relegation of nonlinear dynamics to wave tails, allow a direct and versatile analytical prediction of self-trapping existence conditions and simple soliton scaling laws, which we confirm experimentally in saturated-Kerr self-trapping observed in photorefractives. This approach provides the means to correctly evaluate beam tails in the saturated regime, which is instrumental in the prediction of soliton interaction forces.

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In contrast with classical soliton theory, optical spatial solitons do not generally go hand in hand with integrability,[1–3] as the larger portion of observed nonlinear waves has no explicit analytical description. Whereas this fact does not invalidate the conceptually useful picture of solitonlike mechanics, which finds its sole validity in physical behavior, it does considerably hamper our ability to predict phenomenology. For example, to experimentally generate a self-trapped beam, whether it be to study some peculiar effect or to develop some useful device, we are forced to rely on numerical integration to find the appropriate launch parameters of the wave that most resemble those of the soliton wave, i.e., the so-called existence conditions, which normally translate into a (mathematically unknown) relationship between the wave intensity and width.[4–6] Moreover, we are not able to formulate scaling laws, which are the clues to understanding soliton self-similarity.[7] Finally, and perhaps most importantly, we have no explicit means of evaluating interaction potentials through wave–tail overlap integrals.

The major source of nonintegrability and absence of closed-form solutions for optical spatial solitons is associated with saturation, and the most general embodiment of saturation is the so-called saturated Kerr-like nonlinearity. This nonlinearity is the effective model that describes screening slab solitons in photorefractive crystals,[4–6] one of the more studied breeds of spatial self-trapped beams, and, for example, spatial solitons in atomic vapor[8] and in semiconductor gain media.[9]

Yet, reflecting on known phenomenology, and in particular, on the vast quantity of experiments associated with photorefractive crystals,[6,10,11] we are faced with solitons that have a consistently regular scaling of soliton intensity and width,[7] in the very asymptotic regime where saturation does not allow for closed-form solutions. In this Letter we find what we believe to be the fundamental key to the interpretation and prediction of these asymptotic traits, formulating an analytical description of soliton existence curves, scaling laws, and tail wave segments whose validity rests on the very notion of saturated response.

In contrast with all previous theoretical descriptions,[6,7,12] the heart of our approach is to address the physical implications of saturation. Saturation allows the separation of the nonlinear problem into two distinct regimes: a highly nonlinear and a purely linear one. We address different portions of the same nonlinear optical wave in the transverse plane differently, an approach that proves meaningful when longitudinal dynamics are absent, as for solitons. The scheme that we present, which is based on the matching of linear and nonlinear beam solutions of the correct physical description of the trapping mechanism, is not characterized by any simplifying hypothesis, such as the introduction of an effective threshold linearity,[13] an *a posteriori* fitting procedure, or a Taylor expansion of the saturated nonlinearity,[7] that does not allow a direct prediction of asymptotics. Our scheme manifests a striking predictive power that we discuss here.

Nonlinear saturated beam dynamics can be generally associated with an intensity-dependent index of refraction change, $\Delta n(I)$, that saturates, i.e., $\Delta n(I) \simeq \Delta n_\infty$ (independently of I) for a strong enough optical intensity $I \gg I_s$, where I_s is a characteristic saturation intensity. For beam peak intensity $I_0 \gg I_s$, the beam dynamics can be separated into two parts. One is in proximity to the beam peak, where the nonlinearity is strongly saturated and the index of refraction is not affected by the shape of the optical beam, taking on the limiting value Δn_∞. There being no self-action, in this region beam dynamics are linear. The other is located at the edges of the beam, where saturation is progressively less pronounced and the tails suffer highly nonlinear dynamics. Saturation asymptotically relegates beam nonlinear dynamics to the beam tails.

Translation of this picture into a predictive tool starts from the description of saturated solitons in a scalar 1 + 1-dimensional reduction of the nonlinear parabolic wave equation for the slowly varying envelope $A(x, z)$ of the optical field,

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$E_{\mathrm{opt}}(x,z,t) = A(x,z)\exp(ikz - i\omega t)$, where x is the transverse coordinate, z is the longitudinal beam propagation coordinate, $k = 2\pi n/\lambda$ is the optical wave vector of the λ wavelength field, $\omega = 2\pi c/n\lambda$, and n is the unperturbed index of refraction of the medium. Nonlinearity is mediated by an index modulation Δn that is affected by the optical beam through the intensity distribution $I(x,z) = |A(x,z)|^2$. By imposing the condition that the optical field gives rise to a solitonic propagation of the type $A(x,z) = \sqrt{I_s}\, e^{i\Gamma z} u(\xi)$, where Γ is the soliton eigenvalue, and $\xi = x/d$, with $d = [2k^2|\Delta n(0)|/n]^{-1/2}$, we find that the soliton supporting nonlinear equation becomes

$$u''(\xi) = -\delta u(\xi) + f(|u|^2)u(\xi)\,, \tag{1}$$

where $\delta = -\Gamma/[k\Delta n(0)/n]$ is associated with the boundary conditions[4] and $\Delta n(I)/\Delta n(0) \equiv f(|u|^2)$. In the hypothesis that the nonlinearity is *local,* Eq. (1) has a first integral that allows the direct expression of δ as a function of boundary conditions. For a fundamental bright soliton, $u(0) = u_0$, $u'(0) = 0$, $u(\infty) = u'(\infty) = u''(\infty) = 0$, and $u(\xi)$ can be taken to be real. As a consequence $\delta = (1/u_0^2)\int_0^{u_0^2} f(u^2)d(u^2)$. This equation suggests an analytical expression, once f is a known integrable function that describes the nonlinearity.[4] The basis for this explicit expression for δ depends *both* on the nontrivial explicit formulation of the model and on the assumption that fundamental bright solitons *exist.* As a consequence of this, the expression for δ cannot serve as a prediction of solitons in themselves but only as a subset of asymptotic traits.

The scaling of Eq. (1) in the saturated regime of $u_0 \gg 1$ [i.e., $I(0) \gg I_s$] for the *portion* of the wave in proximity to the peak, where the more general condition $u(\xi) \gg 1$ holds, are such that $f(u^2)_{\max}/\delta \to 0$ for the general case of $f(u^2) = 1/(1 + u^2)^m$ $(m = 1, 2, \ldots)$. In this regime we obtain the simple yet remarkable result that for the saturated portion of the nonlinear response Eq. (1) asymptotically leads to the harmonic oscillator $(\delta > 0)$

$$u''(\xi) + \delta u(\xi) = 0\,. \tag{2}$$

Note that this reduction is valid only once z dynamics have been eliminated, something that is intrinsically incompatible with localized optical waves in linear propagation. Therefore, *the very emergence of the harmonic wave is a product of nonlinearity,* as transpires, for example, if one notes that the elastic constant associated with δ results from the partial integration of the fully nonlinear equation (1).

The main implication of Eq. (2) is that, for $u(\xi)$ to be compatible with bright soliton boundary conditions, it must be of the form $u(\xi) = u_0\cos(\delta^{1/2}\xi)$. The existence conditions are defined as the beam intensity FWHM, $\Delta\xi$, as a function of the root of the beam peak intensity, u_0. In conditions of strong saturation, when both $I(0) \gg I_s$ and $I(0)/2 \gg I_s$, the portion of the beam that is harmonic extends *beyond* the half-width, and thus the existence conditions can be directly evaluated, the result being that

$$\Delta\xi = (\pi/2)\delta^{-1/2}. \tag{3}$$

For Kerr-saturated nonlinearities of the type $f(u^2) = 1/(1 + u^2)^m$, with $m \ge 2$, $\delta \simeq (m-1)^{-1}u_0^{-2}$, and thus Eq. (3) gives asymptotically $\Delta\xi = (\pi/2)(m-1)^{1/2}u_0$. The existence curve is *linear,* thus greatly increasing the predictive power of the whole scheme.

For the fundamental Kerr-saturated case $f(u^2) = 1/(1 + u^2)$, i.e., $m = 1$, $\delta = u_0^{-2}\ln(u_0^2 + 1)$ and Eq. (3) gives $\Delta\xi = (\pi/2)u_0[\ln(u_0^2 + 1)]^{-1/2}$, which is quasi-linear for a given set of values of u_0, given the logarithmic nature of the nonlinear distortion. However, in this case, convergence from Eq. (1) to Eq. (2) is *logarithmic,* and for all practical purposes (i.e., for values of u_0 up to 10^2) $f(u^2)_{\max}/\delta$ does not converge to zero. It is therefore necessary to directly impose the known boundary condition $u''(0) = -\delta u_0 + f(u_0^2)u_0$ onto the harmonic-wave segment from Eq. (1). This amounts to taking the saturated value $f(u_0^2)u$ for $f(u^2)u$ in Eq. (1) instead of wholly neglecting it and clearly requires no knowledge of $u(\xi)$. The resulting corrected elastic constant in Eq. (2) is thus given by $\delta = u_0^{-2}[\ln(u_0^2 + 1) - u_0^2/(1 + u_0^2)]$. The very same boundary matching can be imposed in the $m \ge 2$ case, where, however, convergence is much faster, giving a corrected $\delta = u_0^2(1 + u_0^2)^{-2}$ instead of $\delta = u_0^{-2}$ (see curve and caption of Fig. 1) for $m = 2$.

The scaling laws that allow for self-similarity and fractals are associated with the harmonic-wave segments. In particular, for the $m \ge 2$ case [for which $u(\xi) = u_0\cos(u_0^{-1}\xi)$], the segments obey the self-similar relationship $A(x,z) \to q^{-1}A(qx, q^2 z)$. In the $m = 1$ case {for which $u(\xi) = u_0\cos[u_0^{-1}\ln(u_0^2 + 1)^{1/2}\xi]$}, a rescaling symmetry can be found for only a limited range of scales (where $\delta \propto u_0^2$ is approximately valid), giving the same $A(x,z) \to q^{-1}A(qx, q^2 z)$.

Wave segmentation can be used to predict beam tail structure, which is instrumental in the evaluation of long-range interaction potentials through wave overlap integrals.[14,15] In particular, one can predict when the exponentially decaying regime will supersede the harmonic segment, thus evaluating the

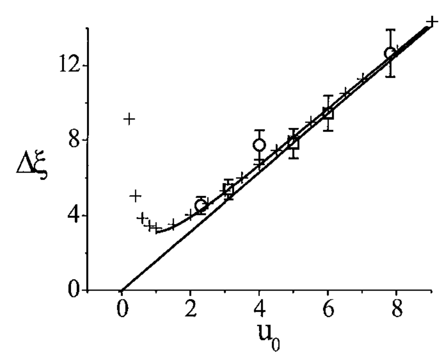

Fig. 1. Segmented wave harmonic theory (solid line and curve) of centrosymmetric screening solitons, numerical integration of Eq. (1) (plus symbols), and experiments in a sample of potassium lithium tantalate niobate (circles and squares). The line is the basic segmentation prediction $\Delta\xi = (\pi/2)u_0$; the curve is the prediction Δ that adheres more closely to the result of $\Delta\xi = (\pi/2)(1 + u_0^2)/u_0$ (see text). There are no free parameters.

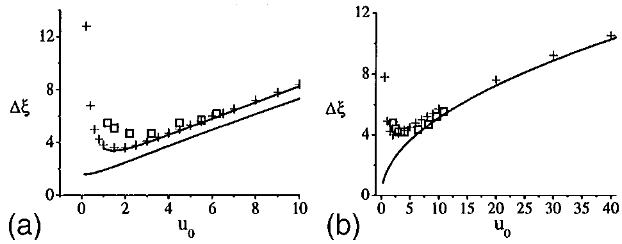

Fig. 2. Segmented wave harmonic theory (curves) of (a) conventional screening solitons and (b) high-intensity solitons. Plus symbols, numerical integration of Eq. (1); squares, experiments in strontium barium niobate.[10,17] In (a) the bottom curve is the basic segmentation prediction $\Delta\xi = (\pi/2)u_0[\ln(u_0^2 + 1)]^{-1/2}$; the top curve is the more adherent prediction $\Delta\xi = (\pi/2)u_0[\ln(u_0^2 + 1) - u_0^2/(1 + u_0^2)]^{-1/2}$. There are no free parameters.

effective transverse shift in the beam tail $\overline{\xi}$ that is due to saturation. The unsaturated tail forms at the edges of the beam where $u \ll 1$, i.e., in proximity to the region where $\delta^{1/2}\xi \simeq \pi/2$, giving $\overline{\xi} \simeq \delta^{-1/2}\pi/2$. The tails, which obey the unsaturated form of Eq. (1) $u'' = -(\delta - 1)u - mu^3$, are therefore described by $u \simeq [2(1 - \delta)/m]^{1/2} \operatorname{sech}[(1 - \delta)^{1/2}(\xi - \overline{\xi})]$. Comparison with numerical saturated waveforms confirms this prediction, passing, for example, in the $m = 2$ case, from an error of $\Delta\overline{\xi}/\overline{\xi} \approx 0.05$ at $u_0 = 10$ to less than 0.005 at $u_0 = 100$.

We pit our asymptotic theory against screening solitons in potassium lithium tantalate niobate, which are described by a saturated Kerr response with $m = 2$, when dielectric nonlinearity is absent.[12,16] In this case we should obtain $\Delta\xi = (\pi/2)u_0$. Our comparison is twofold: the first, a confirmation of the mathematical validity of the wave-segmentation approach, is a comparison of our predictions with those obtained through numerical integration of Eq. (1) with $f(u^2) = 1/(1 + u^2)^2$. The results are shown in Fig. 1. As can be seen, the convergence is strikingly accurate. We obtain a more profound comparison by pitting our predictions against experiments. To have a simple saturated Kerr nonlinearity we heated our samples of potassium lithium tantalate niobate above the Curie temperature, T_c, to a region in which dielectric nonlinearity[11,16] has a negligible effect. In our copper- and vanadium-doped 2.6 mm × 1.8 mm × $6.4^{(z)}$ mm sample, with $T_c = 18\,°\mathrm{C}$, dielectric nonlinearity is absent for values of $T \geq 30\,°\mathrm{C}$. To be sure that no higher-order effects were present, we measured the soliton existence points for two different values of T [$T_1 = 30\,°\mathrm{C}$ (squares) and $T_2 = 35\,°\mathrm{C}$ (circles) in Fig. 1]. As shown in Fig. 1, a single consistent linear signature is observed, consistent both with numerical integration and, most importantly, with our harmonic segmentation theory, which has no free fitting parameters. The uncertainty in the data is due to the precision in the evaluation of the correct value of voltage V applied in the x direction on the crystal sample electrodes for which self-trapping is observed. The results are to our knowledge the first experimental indication of an accessible fractal-supporting soliton system ($m = 2$).

Next, we pit our segmented wave theory against numerical and experimental results relative to the lowest-order saturated Kerr nonlinearity associated with a value of $m = 1$. We consider experiments with strontium barium niobate reported, for example, in Ref. 17, and again compare our analytical predictions in Fig. 2(a).[18]

Solitons described by a weakly saturated Kerr model with $m = 1/2$ have also been reported in photorefractives[17] in the high-beam-intensity regime. For these solitons saturation is so weak that our general treatment cannot be applied. However, wave segmentation indicates that Eq. (1) converges to $u'' + (2/u_0)u - 1 = 0$, which leads to an asymptotic $\Delta\xi = \sqrt{2}\arccos(\sqrt{2} - 1)u_0^{1/2}$ behavior, in its most basic formulation [see comparison of calculations and experiments in Fig. 2(b)].

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18. One can appreciate the difference between the Taylor expansion approach and ours by comparing our Fig. 2(a) and Fig. 1 of Ref. 7.